

RADIATION

Planck's Radiation Formula

Wien's Law and Rayleigh-Jeans' Law for black body radiation, are failed to explain energy distribution of all wavelengths. Wien's law is able to explain energy distribution of shorter wave length range and Rayleigh-Jeans' law is able to explain energy distribution of longer wavelength range experimentally.

Max Planck introduced correct black body radiation formula to explain energy distribution among the various wavelengths on the base of a new theory, called quantum mechanics. He assumed that atoms of cavity ($\alpha_2\alpha_1$) radiator behave as electrical oscillators of all frequencies. These oscillators emit energy (electromagnetic radiation energy) into the cavity and also absorb the same from it. Planck made following assumptions regarding these oscillators—

(i) Any oscillator has discrete energy

$$E = nh\nu$$

where n ~~is~~ $=$ Planck's const.

ν = Freq. of the oscillators

n = An integer number.

(ii) The oscillators ~~do not~~ emit or absorb energy continuously. An oscillator emits or absorbs energy in packets or quanta.

Each packet of has energy $= h\nu$

Let N = Total no. of oscillators

E = Their total energy

Average energy of an oscillator $\epsilon = E/N$.

Value of ϵ is to be determined.

Now, let $N_0, N_1, N_2, \dots, N_r$ be the number of oscillators having the energies $0, h\nu, 2h\nu, \dots, rh\nu$ respectively

A/c to Maxwell's law, no. of oscillators having energy $rh\nu$ at temp T, $N_r = N_0 e^{-rh\nu/KT}$

$$\therefore N = N_0 + N_1 + N_2 + N_3 + \dots$$

$$= N_0 + N_0 e^{-h\nu/KT} + N_0 e^{-2h\nu/KT} + \dots$$

$$= N_0 (1 + e^{-hv/RT} + e^{-2hv/RT} + e^{-3hv/RT} \dots)$$

$$\text{Let } e^{-hv/RT} = y$$

$$N = N_0 (1 + y + y^2 + y^3 + \dots)$$

$$= N_0 \times \frac{1}{1-y}$$

$$N = \frac{N_0}{(1 - e^{-hv/RT})} \quad \text{(1)}$$

* Total energy of the oscillators

$$\begin{aligned} E &= N_0 \times 0 + N_1 h\nu + N_2 \times 2h\nu + N_3 \times 3h\nu + \dots \times r \times rh\nu \\ &= 0 + h\nu N_0 e^{-hv/RT} + 2h\nu N_0 e^{-2hv/RT} + \dots + rh\nu N_0 e^{-rhv/RT} \\ &\Rightarrow N_0 h\nu (e^{-hv/RT} + 2e^{-2hv/RT} + \dots + re^{-rhv/RT}) \\ &= N_0 h\nu (e^{-hv/RT} (1 + 2e^{-hv/RT} + 3e^{-2hv/RT} + \dots)) \\ &= N_0 h\nu e^{-hv/RT} (1 + 2y + 3y^2 + \dots + ry^{r-1}) \\ &= N_0 h\nu e^{-hv/RT} \times \frac{1}{(1-y)^2} \end{aligned}$$

$$\text{or } E = N_0 e^{-hv/RT} \frac{h\nu}{(1 - e^{-hv/RT})^2} \quad \text{(2)}$$

Hence, average energy of an oscillator (eqn (2)/(1))

$$\bar{E} = E/N$$

$$= N_0 e^{-hv/RT} \frac{h\nu}{(1 - e^{-hv/RT})^2} \times \frac{(1 - e^{-hv/RT})}{N_0}$$

$$= \frac{h\nu e^{-hv/RT}}{1 - e^{-hv/RT}} = \frac{h\nu e^{-hv/RT}}{e^{hv/RT} - (1/e^{hv/RT} - 1)}$$

$$\text{or, } \bar{E} = \frac{h\nu}{e^{hv/RT} - 1} \quad \text{(3)}$$

Number of oscillators ~~per unit volume~~ having freq. ν in the freq. range $\nu + (\nu + d\nu)$

$$N = \frac{\delta T D^2}{c^3} d\nu$$

Hence, energy density (energy per unit volume) in the freq. range $\nu + (\nu + d\nu)$

$E d\nu = \text{No. of oscillators/volume} \times \text{Average energy of an oscillator}$

$$= \frac{8\pi\nu^2}{c^3} \times \bar{E} \times d\nu$$

$$= \frac{8\pi\nu^2}{c^3} \frac{h\nu}{(e^{hv/RT} - 1)} d\nu$$

$$\left[\frac{1}{\lambda^5} \cdot \frac{8\pi h c}{(e^{h\lambda/kT} - 1)} d\lambda \right] = \dots \quad (4)$$

This is Planck's radiation formula in terms of freq.

$$E_{dd} = \frac{8\pi h c}{d^5} \frac{d\lambda}{(e^{h\lambda/kT} - 1)}$$

Hence, σ (Noyes density) in the wavelength range

$$E_{dd} = \frac{8\pi h c}{d^5} \frac{d\lambda}{(e^{h\lambda/kT} - 1)}$$

[As λ increases
wavelength density

$$\sigma \left| E_{dd} = \frac{8\pi h c}{d^5} \frac{d\lambda}{(e^{h\lambda/kT} - 1)} \right|$$

This is Planck's radiation formula in terms of wavelength

Wien's law from Planck's law -

A/c to Planck's radiation formula

$$E_{dd} = \frac{8\pi h c}{d^5} \frac{d\lambda}{(e^{h\lambda/kT} - 1)}$$

When d is small then $e^{h\lambda/kT}$ would be large, \approx incorporation to 1.

Above eqn becomes

$$\begin{aligned} E_{dd} &= \frac{8\pi h c}{d^5} \frac{d\lambda}{e^{h\lambda/kT}} \\ &= \frac{8\pi h c}{d^5} e^{-h\lambda/kT} d\lambda \end{aligned}$$

$$\text{let } 8\pi h c = A \text{ & } \frac{hc}{k} = B$$

$$\therefore E_{dd} = \frac{A}{d^5} \frac{1}{e^{B/\lambda T}} d\lambda$$

$$\sigma | E_{dd} = A d^{-5} e^{-B/\lambda T} d\lambda$$

This is Wien's law holding good at short wavelengths

Rayleigh-Jeans law from Planck's law -

A/c to Planck's radiation formula

$$E_{dd} = \frac{8\pi h c}{d^5} \frac{d\lambda}{(e^{h\lambda/kT} - 1)}$$

when d is large then $e^{h\lambda/kT} \ll 1$

$$e^{\frac{hc}{\lambda K T}} = 1 + \frac{hc}{\lambda K T} + \dots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \dots$$

$$\therefore E_{dd} = \frac{8\pi h c}{\lambda^5 \left(1 + \frac{hc}{\lambda K T} + \dots \right)} d\lambda$$

$$\text{or } E_{dd} = \frac{8\pi h c}{\lambda^5 \frac{hc}{\lambda K T}} d\lambda$$

$$\text{or } \boxed{E_{dd} = \frac{8\pi K T}{\lambda^4} d\lambda}$$

This is Rayleigh-Jeans Law which holds good at long wavelength.

Q

Forced vibration or Simple harmonic motion—

When a vibrating body vibrates with the help of external periodic force then such vibration is called forced vibration. Forced vibration can be obtained for a long time.

Theory — Consider damped vibration of a particle of mass m .

(i) Two damped forces act on the particle —

Restoring force — This force is directly proportional to displacement y of the particle at any time. Hence,

Restoring force = $-ky$, where k is a constant. — ve sign indicates that restoring force is opposite to y .

(ii) Frictional force — This force is directly proportional to Velocity dy/dt of the particle at any time.

Hence, ~~Restoring force~~ frictional force = $-r dy/dt$ where r is a constant.

At displ. y , acceleration of the particle = d^2y/dt^2

Hence, force acting on the particle = $m \cdot d^2y/dt^2$

Hence, $m \frac{d^2y}{dt^2} = -r \frac{dy}{dt} - ky$

$$m \frac{d^2y}{dt^2} + r \frac{dy}{dt} + ky = 0$$

This represents eqn of damped S.H.M. (vibration)

$$\frac{d^2y}{dt^2} + \frac{r}{m} \frac{dy}{dt} + \frac{k}{m} y = 0$$

$$\text{Let } \frac{r}{m} = 2K \quad \text{and } \frac{k}{m} = \omega^2$$

where K = damping coeff.

Above eqn becomes

$$\frac{d^2y}{dt^2} + 2K \frac{dy}{dt} + \omega^2 y = 0 \quad (1)$$

Let external periodic force $F \sin \omega t$ is applied on the particle.

Above eqn. becomes

$$\frac{dy}{dt^2} + 2K \frac{dy}{dt} + \omega^2 y = F \sin \omega t \quad (2)$$

where F = applied force/mass & $\frac{1}{2\pi} = \text{Freq. of the periodic force}$.

This represents diff. eqn. of forced vibration. Solution of this eqn. is in two parts —

(i) Complementary function (पूर्ण फलन)

(ii) Particular integral (विशेष उत्तराणि).

(i) Solution of complementary function —

When applied force $F \sin \omega t$ then eqn. of the forced

Vibration

$$\frac{d^2y}{dt^2} + 2K \frac{dy}{dt} + \omega^2 y = 0$$

Solution of this eqn. is the sum of Complementary function.
When $K < \omega$ then sum of the above eqn (as sum of the damped vibration)

$$y = \frac{y_0 e^{-kt}}{\sqrt{\omega^2 - k^2}} \sin\{\sqrt{\omega^2 - k^2} t + \phi_1\} \quad (3) \quad [a = \omega]$$

where $\frac{y_0 e^{-kt}}{\sqrt{\omega^2 - k^2}}$ = amplitude, $\sqrt{\omega^2 - k^2}$ = Arg.vel., $t + \phi_1$ = phase of the vibration.

(ii) Soln. of Particular Integral —

Let soln. of eqn (2) of particular integral

$$y = b \sin(pt - \phi) \quad (4)$$

$$\therefore \frac{dy}{dt} = b p \cos(pt - \phi)$$

$$\therefore \frac{d^2y}{dt^2} = -b p^2 \sin(pt - \phi)$$

From eqn (2)

$$\begin{aligned} -b p^2 \sin(pt - \phi) + 2K b p \cos(pt - \phi) + \omega^2 b \sin(pt - \phi) &= F \sin pt \\ &= F \sin[(pt - \phi) + \phi] \\ &= F [\sin(pt - \phi) \cos \phi + \cos(pt - \phi) \sin \phi] \\ &= F \sin(pt - \phi) \cdot \cos \phi + F \cos(pt - \phi) \cdot \sin \phi \end{aligned}$$

Equating coeff. of $\sin(pt - \phi)$ & $\cos(pt - \phi)$ from both sides

$$-b p^2 + \omega^2 b = F \cos \phi$$

$$\therefore b(\omega^2 - p^2) = F \cos \phi \quad (i)$$

$$+ 2K b p = F \sin \phi \quad (ii)$$

Squaring & adding

$$b^2 (\omega^2 - p^2)^2 + 4K^2 b^2 p^2 = F^2$$

$$\therefore F^2 = b^2 \{(\omega^2 - p^2)^2 + 4K^2 p^2\}$$

$$\therefore F = b \sqrt{(\omega^2 - p^2)^2 + 4K^2 p^2}$$

$$\therefore b = \frac{F}{\sqrt{(\omega^2 - p^2)^2 + 4K^2 p^2}} \quad (5)$$

$$\text{Eqn. (i)/(ii)} \quad \tan \phi = \frac{2Kp}{(\omega^2 - p^2)} \quad (6)$$

Hence, Complete soln. of eqn. (2) from eqn (3) & (4)

$$y = \frac{y_0 e^{-kt}}{\sqrt{\omega^2 - k^2}} \sin\{\sqrt{\omega^2 - k^2} t + \phi_1\} + b \sin(pt - \phi) \quad (7)$$

Values of b &. ϕ are obtained from eqn (5) & (6)

First part of eqn (7) represents damped vibration having freq. $\sqrt{\omega^2 - k^2}$ & amplitude $\frac{y_0 e^{-kt}}{\sqrt{\omega^2 - k^2}}$ whose value decreases with time & at last becomes zero. At last after some time eqn. of the forced vibration is represented by

$$y = b \sin(pt - \phi)$$

This represents Simple harmonic vibration having amplitude b & freq. $\frac{1}{2\pi}$. Value of b does not change with time & freq. $\frac{1}{2\pi}$ is equal to the freq. of the applied force. ϕ is the phase diff. between the vibration and applied periodic force.

Amplitude of forced vibration and resonance —

Amplitude of the forced vibration

$$b = \frac{F}{\sqrt{(\omega^2 - \beta^2)^2 + 4K^2\beta^2}}$$

Value of b depends upon the values of ω & β .

For maximum amplitude $\omega^2 - \beta^2 = 0$

$$\text{or, } \omega = \beta$$

$$\therefore b_{\max} = \frac{F}{\sqrt{4K\beta^2}} \\ = \frac{F}{2K\beta}$$

Henry, when natural freq (ω) of the body is equal to the freq. (β) of the applied force then amplitude is maximum. This condition of vibration is called resonance.

In case of resonance if damping $K=0$

$$\text{then amplitude } b = F/\alpha = \infty$$

In practice this is not possible because no medium is dampingless.

Q What is forced vibration? Discuss the vibration of a particle executing damped harmonic motion when subjected to an external periodic force. When does resonance occur.

Wander law of state form

clausius deduced the eqⁿ of an imperfect gas more rigorously. The gaseous Particles are treated as ~~perfect~~ ordinary matter subject to classical dynamics. They if x, y, z denote the coordinates of the centre of any molecule measured with respect to a certain system of axes and F_x, F_y, F_z denote the components of the force acting on the molecule. Then

$$F_x = m \frac{d^2 x}{dt^2}, \quad F_y = m \frac{d^2 y}{dt^2}, \quad F_z = m \frac{d^2 z}{dt^2}$$

for translational motion of gaseous particles we know that

$$\frac{1}{2} m \dot{x}^2 = \frac{m}{4} \frac{d^2}{dt^2} (\dot{x}^2) - \frac{1}{2} F_x \cdot x \quad (1)$$

similarly for other coordinates

$$\frac{1}{2} m \dot{y}^2 = \frac{m}{4} \frac{d^2}{dt^2} (\dot{y}^2) - \frac{1}{2} F_y \cdot y \quad (2)$$

$$\text{and } \frac{1}{2} m \dot{z}^2 = \frac{m}{4} \frac{d^2}{dt^2} (\dot{z}^2) - \frac{1}{2} F_z \cdot z \quad (3)$$

adding eq (1) (2) and (3) we get

$$\frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = \frac{m}{4} \frac{d^2}{dt^2} (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - \frac{1}{2} (F_x \cdot x + F_y \cdot y + F_z \cdot z)$$

which gives the kinetic energy of a molecule.

Now summing this expression over all molecules of the gas and average over a long time T is taken then we get.

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$$\frac{1}{2} \overline{\sum m c^2} = \frac{m}{t} \cdot \overline{\sum \frac{d^2}{dt^2} (\vec{r}^2)} - \frac{1}{2} \overline{\sum (F_x \cdot x + F_y \cdot y + F_z \cdot z)}$$

Where the summation Σ extends over all molecules and the bar denotes time average from 0 to t . If the gas is in a steady state the quantities $\frac{1}{2} \sum m c^2$ and $\sum F_x \cdot x + F_y \cdot y + F_z \cdot z$ are approximately constant throughout the motion. The second term in eqn (5) is equal to

$$\frac{m}{t} \int_0^t \sum \frac{d^2}{dt^2} (\vec{r}^2) dt = \frac{m}{t} \int_0^t \left[\frac{d}{dt} (\vec{r}^2) \right] dt$$

Now $\frac{d}{dt^2} (\vec{r}^2) = 2 \vec{r} \frac{d\vec{r}}{dt}$ and $\frac{d\vec{r}}{dt}$ for a molecule fluctuates irregularly with time being often positive as negative, while ' r ' from the theory of Brownian movement varies as the square root of time. Hence by making t large the term $\int_0^t \left[\frac{d}{dt} (\vec{r}^2) \right] dt$ becomes zero and eqn (5)

reduces to:

$$\frac{1}{2} \sum m c^2 = - \frac{1}{2} \sum (F_x \cdot x + F_y \cdot y + F_z \cdot z) \quad (6)$$

The expression on the right is called the virial of the system.

Eqn (6) asserts that the total kinetic energy of translation of the molecules of a gas in the steady state is equal to its virial which is the virial theorem of Clausius.

Now F_x , F_y and F_z denote all the forces which exert upon molecules and must find the contribution of all such forces to the virial. These forces may be broadly divided into the following three categories:-

- (i) The force of impact on the molecule due to collision with the walls of the containing vessel
- (ii) The forces due to collision between two molecules
- (iii) The forces of attraction between molecules when separated by some distance.

Let us calculate the contribution of (i) to the virial. Let the gas is confined in a parallelepiped with sides a , b , c parallel to the coordinate axes and while the two faces cut the x -axis at x_1 , x_2 ($\therefore x_2 - x_1 = a$) If the pressure exerted by the gas is p , the value of $\sum F_{xi}$ at x_1 is pbc and at x_2 is $-pbc$.

$$\therefore -\frac{1}{2} \sum F_{xi}x_i = \frac{1}{2} pbc(x_2 - x_1) = \frac{1}{2} p \cdot abc = \frac{1}{2} pV$$

Where V is the volume of parallelepiped. Similarly for the other pairs of faces. Hence the total contribution by the forces acting at the walls is $\frac{3}{2} pV$.

[If we assume the gas to be perfect, the forces mentioned in (ii) do not exist and since the size of the molecule is negligible for a ,

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GE

Perfect gas the contributions due to forces in (2) & (3) are zero and the relation are equal and of opposite sign and in this case they also act at the same place. Hence for a perfect gas we have from the theorem of virial

$$\frac{1}{2} m \dot{r}^2 = \frac{3}{2} PV - \frac{1}{3} m d^2 \quad (1)$$

For a real gas, the effect of forces acting between the molecules of the gas including the forces during collision ~~between~~ are taken into account assuming that the molecules have finite size. The nature of forces is not definitely known.

for present let us assume that the force between two molecules is radial and is equal to $\phi(r)$ where r is the distance between the molecules.

If it is assumed to depend only upon the intermolecular distance not upon the orientation of the molecules $\phi(r)$ will be considered positive when it is repulsive.

Evidently $\phi(r) = -\frac{dE}{dr}$, where E is the potential energy due to intermolecular forces.

Let the co-ordinates of the force centres of the two molecules be x_1, y_1, z_1 and x_2, y_2, z_2 respectively. Then the component of F_x, F_y, F_z and F'_x, F'_y and F'_z acting on them will be given by

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$$F_x = \phi(r) \frac{(x-x_1)}{r}, \quad F_x' = \phi(r) \frac{(x-x_1')}{r}$$

The Contribution to $\sum F_{xi} x_i$ made by these forces. 4

$$F_x \cdot x + F_{x'} \cdot x' = \phi(r) \frac{(x-x_1)^2}{r^2} \quad (8)$$

Similarly for other components.

Hence the total contribution by the intermolecular forces to the virial is

$$-\frac{1}{2} \sum \frac{\phi(r)}{r} \left\{ (x-x_1)^2 + (y-y_1)^2 + (z-z_1)^2 \right\} \\ = -\frac{1}{2} \sum r \phi(r)$$

where the summation extends over the each pair of molecules counted once. Hence the virial theorem yields

$$\frac{1}{2} m \bar{v}^2 = \frac{3}{2} PV - \frac{1}{2} \sum r \phi(r) \quad (9)$$

Since the mean K.E. of translation.

$$\frac{1}{2} m \bar{v}^2 = \frac{3}{2} kT. \text{ So eqn } (9) \text{ yields.} \\ PV = n kT - \frac{1}{2} \sum r \phi(r) \quad (10) \text{ out where } k \text{ is Boltzmann constant}$$

which is the general form

of eqn of state.

Now from Maxwell Boltzmann law

$$n_e = n_0 \left(\frac{m}{2\pi kT} \right)^{3/2} e^{-\frac{(E_m + E)}{kT}} \frac{4\pi r^2 dr}{kT}$$

Where n_e = Number of molecules per c.c. having velocity lying between r and $r+dr$ and having the potential energy E . n_0 is the molecular density in regions where the potential energy is zero. Integrating for all values of r from 0 to ∞ we get

$$n_e = n_0 e^{-E/kT} \quad (11)$$

Let us calculate the contribution of one molecule to the virial. The average number of other molecules lying within a distance r for dr from this molecule is $n \pi r^2 dr$ and for those the force is $\phi(r)$. Hence the contribution of one molecule to $r\phi(r)$ is

$$4\pi r^3 n \bar{e}^{-Er/RT} \phi(r) dr$$

and the contribution to N molecules of the gas is $\frac{N}{2} \int_0^\infty 4\pi r^3 n \bar{e}^{-Er/RT} \phi(r) dr$ — (12)

Putting $n = \frac{N}{V}$ in the correction term then the eqn (10) yields

$$PV = NKT + \frac{2\pi}{3} \frac{d^2}{V} \int_0^\infty r^3 \bar{e}^{-Er/RT} \phi(r) dr — (13)$$

Integrating eqn (13) by parts we get

$$\begin{aligned} PV &= NKT + \frac{2\pi}{3} \frac{d^2}{V} \left[\left\{ r^3 KT \bar{e}^{-Er/RT} \right\}_0^\infty \right. \\ &\quad \left. - \int_0^\infty r^2 (KT \bar{e}^{-Er/RT}) dr \right] \\ &= NKT + \frac{2\pi d^2}{V} KT \cdot \int_0^\infty r^2 \left(1 - \bar{e}^{-Er/RT} \right) dr — (14) \end{aligned}$$

[since $\phi(r) = -\frac{\partial Er}{\partial r}$ we have
 $\bar{e}^{-Er/RT} \phi(r) dr = KT \bar{e}^{-Er/RT} dr + A$
where A is the constant]

Where A has been put equal to $-KT$ to secure convergence of the integral at infinity since Er being small at infinity the term $\{ \}$ gives $r^3 Er$ which vanishes at infinity as Er is negligible.

Van der Waals made the assumptions that the molecules are rigid spheres of diameter σ and attract each other with a weak force which rapidly decreases as the distance increases. On account of attractive force, E has a small negative value. for $r > \sigma$ the integral in eqn (4) becomes:

$$\int_0^{\sigma} r^2 dr + \int_{\sigma}^{\infty} r^2 \left(1 - e^{-\frac{Er}{RT}}\right) dr = \frac{\sigma^3}{3} + \int_{\sigma}^{\infty} r^2 \frac{Er}{RT} dr$$

then eqn (4) yields

$$PV = RT + \frac{RT}{V} \left(b - \frac{a}{RT}\right) \quad \text{--- (15)}$$

$$\text{where } b = \frac{q}{3} + \pi \sigma^2, \quad a = -2\pi \sigma^2 \int_0^{\infty} r^2 E_r dr \quad \text{--- (16)}$$

$$PV = RT + \frac{RTb}{V} - \frac{q}{V}$$

$$P = \frac{RT}{V} + \frac{RTb}{V^2} - \frac{q}{V^2}$$

$$(P + \frac{q}{V^2}) = \frac{RT}{V} (1 + \frac{b}{V})$$

$$(P + \frac{q}{V^2}) \frac{V}{(1 + \frac{b}{V})} = RT$$

$$\text{or } (P + \frac{q}{V^2}) \frac{V(1 - \frac{b}{V})}{(1 + \frac{b}{V})(1 - \frac{b}{V})} = RT$$

$$(P + \frac{q}{V^2}) \frac{(V-b)}{(1 - \frac{b^2}{V^2})} = RT$$

$$\because \frac{b^2}{V^2} \ll 1 \therefore 1 - \frac{b^2}{V^2} \approx 1$$

$$\boxed{(P + \frac{q}{V^2})(V-b) = RT} \quad \text{--- (17)}$$

which is the van der waals eqⁿ. for a real gas and 'a' and 'b' are called van der waals constants.

Torsional rigidity or modulus of torsion (सरोज हड्डी या हड्डी गुणात्)

Twisting couple per unit angular twist

of a wire or cylinder is called torsional rigidity or modulus of torsion.

Torsional rigidity of a wire or cylinder—

Let a wire (cylinder) of radius r and length l . At upper end $A'B'$ is fixed and twisting couple is applied τ to length at its lower end.

Due to which radius $O'P$ is twisted through an angle θ to the position OPI . Angle θ is called angle of twist.

When the wire is twisted then restoring couple is produced due to elasticity. This restoring couple tends to bring the wire in its original condition.

Due to twist a line CP (parallel to $O'P$) on its surface displaces through an angle ϕ to the position $C'P'$. Angle ϕ is called angle of shear.

To determine torsional couple let the wire is made of several co-axial cylindrical shell. Consider a shell of radius x & thickness dx . As the radius of the shell decreases angle of shear (ϕ) decreases but angle of twist (θ) remains constant. Let ϕ' be the angle of shear of this shell.

$$\text{A/c to fig, } \phi\phi' = l\theta^o \quad (\because \theta = \frac{l\theta}{l})$$

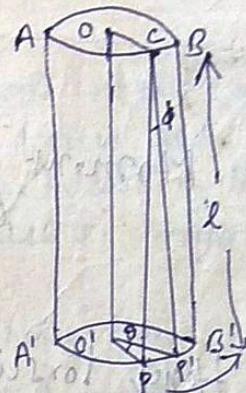
$$\therefore \phi\phi' = x\theta$$

$$\therefore l\theta^o = x\theta$$

$$\therefore \phi' = \frac{x\theta}{l}$$

If η be the Coeff. of rigidity of the wire then

$$\eta = \frac{\text{Shear Stress}}{\text{Shear Strain } \phi'}$$



$$\text{or, } T = \eta \cdot \phi'$$

$$\text{Again, } \text{or, } T = \frac{\eta x \cdot \theta}{l} \text{ by putting the value of } \phi'$$

Area of lower surface of the shell = $2\pi x \cdot dx$

$$\text{Shear force acting on this area, } = T \cdot \text{Area}$$

$$= \frac{\eta x \cdot \theta}{l} \cdot 2\pi x \cdot dx$$

$$= \frac{2\pi\eta\theta}{l} \cdot x^2 \cdot dx$$

$$\text{Moment of couple acting on this shell about the axis } OO' = \frac{2\pi\eta\theta}{l} x^2 dx \cdot x$$

$$= \frac{2\pi\eta\theta}{l} x^3 dx$$

Hence, torsional couple acting on the whole wire

$$T = \frac{2\pi\eta\theta}{l} \int_0^R x^3 dx$$

$$\text{Define girder's width } = \frac{2\pi\eta\theta}{l} \cdot \frac{r^4}{4}$$

$$\text{Total shear force per unit length } = \frac{2\pi\eta\theta r^3}{2l}$$

This is the torsional couple for the origin of twist θ .

Hence, torsional couple per unit angular twist

$$C = \frac{\pi\eta r^4}{2l}$$

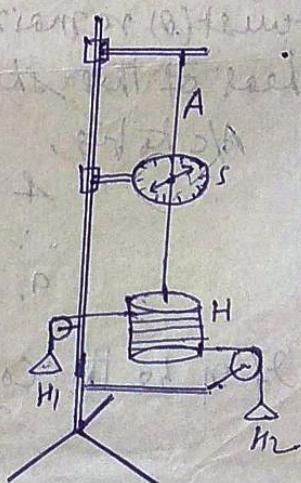
A/c to definition

$$\text{torsional rigidity } C = \frac{\pi\eta r^4}{2l}$$

To determine modulus of rigidity (η) of a wire —
(by statical method)

To determine η of a wire by statical method Barton apparatus is used as shown in the figure.

One end of the given wire is clamped and its lower end end



is clamped at the centre of heavy cylinder H. The wire passes through the centre of a circular scale. Twist produced in the wire is measured by a pointer attached to the wire. Two strings are wound in the same direction over the cylinder and free end of the each string passes over two frictionless pulley in opposite directions. Hangers H_1 & H_2 are suspended from the each free end.

In experiment ~~equal and opposite~~^{some} weights (m) are kept on each hanger so that equal and forces act on the cylinder. Both forces form couple. Due to which the cylinder is rotated & hence the wire is twisted. Angle of twist (θ) is noted on the circular scale.

If d be the diameter of the cylinder Then
torsional couple acting on the wire, $C = mg \cdot d$ — (1)

If η = rigidity modulus, l = length & r = radius of the wire then torsional rigidity of the wire = $\frac{\pi \eta r^4}{2l}$

Since, θ is the angle of twist in the wire ~~then~~ hence
torsional couple acting on the wire $C = \frac{\eta \pi r^4 \theta}{2l}$ — (2)

From eqn (1) & (2),

$$\frac{\eta \pi r^4 \theta}{2l} = mgd$$

or.
$$\eta = \frac{2l mgd}{\pi \theta r^4}$$

From this eqn. modulus of rigidity of the wire may be determined.